Petrsen's Eigenvalue Pinching Theorem

Oct 22nd @ JHU

Introduction:

Goal: Study the shape/structure of spaces with $Ric \geq kg$.

Why? Lower curvature bounds strongly influence the nature of a space through topological, analytic, and geometric consequences. As a bit of moral intuition, recall that for (M^n, g):

(1) **Sectional curvatures**, the information contained in the full Riemann curvature tensor $Rm(x,\vec{v},\vec{w})$, encode the "hessian" data of the metric g: Taylor expanding the metric in normal coordinates at $p \in M$ gives

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3} R_{ijkl}(p) x^k x^l + O(d(x,p)^3).$$

So, (M^n, g) with $sec \geq k$ "are like" u with $D^2 u \geq k\,Id$

(2) **Ricci curvature**, the average of sectional curvature in a given direction, is encoded in a 2-tensor $Ric(X,Y)$ which is a sort of "Laplacian" of the metric g: Expanding in harmonic coordinates at $p \in M$ gives

$$Ric_{ij} = -\frac{1}{2}\Delta_g(g_{ij}) + Q(g^{-1}, \partial g)$$

So, (M^n, g) with $Ric \geq \Lambda$ "are like" u with $-\Delta u \geq \Lambda$

and eg Einstein manifolds "are like" u with $-\Delta u = \Lambda$

So, we should expect comparison results: higher curvature leads to directional inequalities with respect to models ...

(3) **Scalar Curvature?** open questions!

Comparison Geometry for Positive Ricci Curvature:

Goal: Study (M^n, g) with $\mathrm{Ric}_g \geq (n-1)g$ by making comparisons with the model space (S^n, g_{rd}) (which has $\mathrm{Ric}_{g_{rd}} = (n-1)g_{rd}$).

Theorem: (Bishop Gromov '64, '80)

$\mathrm{vol}(M^n, g) \leq \mathrm{vol}(S^n, g_{rd})$ with "=" iff $(M^n, g) \underset{\text{isom}}{\equiv} (S^n, g_{rd})$

Theorem: (Bonnet-Myers – Obata '41, '75')

$\mathrm{diam}(M^n, g) \leq \mathrm{diam}(S^n, g_{rd}) = \pi$ with "=" iff $(M^n, g) \equiv_{\text{isom}} (S^n, g_{rd})$

Theorem: (Lichnerowitz – Obata '58, '62)

Consider the L^2-Laplace spectrum of Δ_g, with

$$\lambda_1(M^n, g) = \inf_{\phi \in C^\infty(M)} \int_M |D\phi|^2 \qquad (\Delta_g f + \lambda f = 0)$$
$$\|\phi\|_2 = 1$$

Then $\lambda_1(M^n, g) \geq \lambda_1(S^n, g) = n$, with "=" iff $(M^n, g) \equiv_{\text{isom}} (S^n, g_{rd})$

Question: Do these rigidity results have effective stable versions?

For BMO and LO, **NO!** In fact, for such (M^n, g),

Cheng-Croke ('82) showed that $\lambda_1(M^n, g) \approx n \leftrightarrow \mathrm{diam}(M^n, g) \approx \pi$

Cheeger-Colding ('96) showed that $\mathrm{diam}(M^n, g) \approx \pi$ iff $(M^n, g) \approx_{GH} (0, \pi) \times_{\sin} X$

where X is a length space of $\mathrm{diam} \leq \pi$.

Anderson ('90), Obata ('91) showed X need not be a sphere, i.e. (M^n, g) need not be S^n topologically. (eg M^n could be $\mathbb{RP}^n$ $(n \geq 2)$ or $S^{n-k} \times S^k$, $(n \geq 2,\ n \geq k+3)$ resp).

(I think topological stability *is* known to be true in $\dim_{\mathbb{R}} = 3$, but can't remember who did it / how... — perhaps Simon's Ricci Flow smoothing...)

Nevertheless,

we can obtain some positive results by considering other invariants:

Def: The radius of a space X is defined by

$$\mathrm{rad}(X) = \inf_{p\in X} \max_{q\in X} d(p,q) = \min\{r>0 : \exists p\in M,\ \overline{B}_p(r)\supseteq X\}$$

Fact: $\mathrm{rad}(X) \le \mathrm{diam}(X)$

Ex: If $X = (0,\pi)\times_{\sin} Z$ with Z a length space of $\mathrm{diam}(Z)\le\pi$,

$$\mathrm{rad}(X) = \max\left\{\tfrac{\pi}{2}, \mathrm{diam}\, Z\right\} \le \pi = \mathrm{diam}\, X$$

eg)
- $X = (S^n_+, g_{\mathrm{rd}})$, $\mathrm{rad}\, X = \frac{\pi}{2} < \mathrm{diam}(X) = \pi$.
- $X = (S^n, g_{\mathrm{rd}})$, $\mathrm{rad}\, X = \pi = \mathrm{diam} X$

As it turns out, the radius does end up being strong enough to control (M^n, g). So too is the volume. In fact, Colding ('96) proved:

Theorem: (Colding '96) (M^n, g) closed, $\mathrm{Ric}_g \ge (n-1)g$. TFAE:

1) $\mathrm{vol}(M^n, g) \approx \mathrm{vol}(S^n, g_{\mathrm{rd}})$
2) $\mathrm{rad}(M^n, g) \approx \pi$
3) $(M^n, g) \approx_{GH} (S^n, g_{\mathrm{rd}})$

} Perelman, Cheeger-Colding, Wang-Zhou: under these conditions, $M^n \approx S^n$ (homeo), (diffeo), (canonical bi Hölder diffeo. with optimal regularity)

Question: Like the passage from $\mathrm{diam}(X)$ to $\mathrm{rad}(X)$ to garner stability, can we obtain stability by considering a stronger spectral assumption?

Let's think about (S^n, g_{rd}) more wisely (?). Consider $(S^n, g_{\mathrm{rd}}) \hookrightarrow \mathbb{R}^{n+1}$.

Then the eigenfunctions of $\Delta_{g_{\mathrm{rd}}}$ are the restrictions to S^n of the homogeneous harmonic polynomials on $\mathbb{R}^{n+1}$. In particular, the first eigenspace for $\lambda_1(S^n, g_{\mathrm{rd}}) = n$ is spanned by the restrictions of the $(n+1)$-coordinate functions $\{x^i|_{S^n}\}_{i=1}^{n+1}$.

Def: Let $f_i(x) = x^i|_{S^n}$, so that $\Delta_{g_{\mathrm{rd}}} f_i + n f_i = 0$ on S^n for $i = 1, \dots, n+1$. In other words, $\lambda_1(g_{\mathrm{rd}}) = n$ has multiplicity $n+1$.

So the key idea is: Instead of just asking for $\lambda_1 \approx n$, we should ask for $\lambda_{n+1} \approx n$ in order to control the whole "first" eigenspace.

Main Theorem (Petersen '99) (M^n, g) closed, $\mathrm{Ric} \geq (n-1)g$

Then $\lambda_{n+1} \approx n$ iff $\mathrm{rad}(M^n, g) \approx \pi$.

Rmk: It turns out that it is sufficient to have $\lambda_n \approx n$ (Aubry). Moreover, $\lambda_k \approx n$, $k = 1, \dots, n$, iff (M^n, g) contains a subset GH close to (S^{k-1}, g_{std}). (Bertrand, Honda).

Background and Set-up:

First of all, let's understand the model space $(S^n, g_{std}) \subseteq \mathbb{R}^{n+1}$:

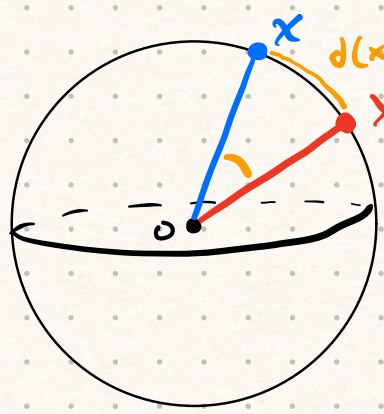
$\{f_i(x) = x^i|_{S^n}\}_{i=1}^{n}$ span the 1st eigenspace for $\lambda_1 = n$

and they satisfy:

1) $\Delta f_i + n f_i = 0$ $\quad \left(\text{note} \Rightarrow \int_{S^n} f_i = 0\right)$

2) $\int_{S^n} f_i^2 = \frac{1}{n+1}$

3) $\int_{S^n} f_i f_j = 0 \qquad i \neq j$



If $d = d_{x_0} = \mathrm{dist}(x_0, -)$ is any distance function on S^n, then

4) $\cos d(x) = \langle x, x_0 \rangle_{euc} = \sum_{i=1}^{n+1} x^i x_0^i = x_0^i f_i(x)$.

(so, the cosine of a distance function is a lin. comb. of 1st eigfctns)

5) $\Delta \cos d + n \cos d = 0$ $\quad$ (by (4))

6) $\int_{S^n} |\cos d|^2 = \frac{1}{n+1}$

7) $\int_{S^n} \cos d = 0$

Moral: cosines of distance functions on S^n (fundamental, geometric) behave similarly to eigenfunctions (more abstract, analytic)

Consequently: We obtain a very useful / key result:.

The identity map $i: S^n \subseteq \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ can be written in several equivalent ways:

(1) $\Phi: S^n \to \mathbb{R}^{n+1}$. $\Phi(x) = (f_1(x), \dots, f_{n+1}(x))$
$= (x^1, \dots, x^{n+1})$
$= x$

(the vector of 1st eigfctns)

(2) $\Psi: S^n \to \mathbb{R}^{n+1}$. $\Psi(x) = \left((n+1) \int_{S^n} f_i(y) \cos d_x(y) \, dy \right)_{i=1}^{n+1}$
$= \left((n+1) \int_{S^n} x^j f_i(y) f_j(y) \, dy \right)_{i=1}^{n+1}$
$= (x^1, \dots, x^{n+1})$
$= x$

(the Fourier coefficients of $\cos d_x(-)$)

Looking ahead: Given some (M^n, g) as in the Main Theorem, we'll study the analogous maps to show that they nearly map to the round sphere from M.

Lastly, our conventions for spectral theory on a general manifold are as follows:

- $\Delta = \operatorname{tr} \nabla^2 = \operatorname{div} \circ \nabla$
- For each (M^n, g), $\exists$ diverging sequence of eigvals $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \to \infty$ listed with multiplicity. The eigfctn eqtn is $\Delta f + \lambda f = 0$.
- We choose an $L^2(M)$ orthog eigenbasis $\{f_i\}_{i=1}^{\infty}$: $\Delta f_i + \lambda_i f_i = 0$ with
  - $\int_M f_i^2 = \frac{1}{n+1}$ $i = 1, 2, \dots$ $f_0 = \frac{1}{\sqrt{n+1}}$.
  - $\int_M f_i = \int_M f_i f_j = \int_M \langle \nabla f_i, \nabla f_j \rangle = 0$ $i \neq j$

- $\int_M |\nabla f_i|^2 = \frac{\lambda_i}{n+1}$

- Let $f \in C^\infty(M)$, and write its Fourier expansion

$$f(x) = \sum_{i=0}^{\infty} \alpha_i f_i \text{, where } \alpha_i = (n+1)\int_M f \cdot f_i$$

Then

$$\int_M |f|^2 = \frac{1}{n+1} \sum_{i=0}^{\infty} \alpha_i^2$$

As a jumping off point for Petersen's Proof, let's recount the proof of the

Obata-Lichnerowicz Theorem: (M^n, g) closed, $\mathrm{Ric} \geq (n-1)g$.
Then $\lambda_1 \geq n$ with "=" iff $(M^n, g) = \text{isom}\,(S^n, g_{rd})$

Proof: **Key Tool**: **Bochner's Formula**: $\forall f \in C^\infty(M)$

$$\tfrac{1}{2}\Delta|\nabla f|^2 = |D^2 f|^2 + \langle \nabla \Delta f, \nabla f\rangle + \mathrm{Ric}(\nabla f, \nabla f)$$

Apply this to the 1st eigfcn f_1:

$$\begin{aligned}
\tfrac{1}{2}\Delta|\nabla f_1|^2 &= |D^2 f_1|^2 - \lambda_1 |\nabla f_1|^2 + \mathrm{Ric}(\nabla f_1, \nabla f_1) \\
&= |D^2 f_1|^2 - \frac{(\Delta f_1)^2}{n} + \frac{(\Delta f_1)^2}{n} - \lambda_1 |\nabla f_1|^2 + \mathrm{Ric}(\nabla f_1, \nabla f_1) \\
&= \left|D^2 f_1 - \frac{\Delta f_1}{n} g\right|^2 + \frac{\lambda_1^2}{n} f_1^2 - \lambda_1 |\nabla f_1|^2 + \mathrm{Ric}(\nabla f_1, \nabla f_1) \\
&\geq \frac{\lambda_1^2}{n} f_1^2 - \lambda_1 |\nabla f_1|^2 + (n-1)|\nabla f_1|^2 \\
&= \frac{\lambda_1^2}{n} f_1^2 + (n-1-\lambda_1)|\nabla f_1|^2
\end{aligned}$$

Integrate over M:

$$0 \geq \int_M \left(\frac{\lambda_1^2}{n} f_1^2 + (n-1-\lambda_1)|\nabla f_1|^2\right) = \frac{\lambda_1^2}{n(n+1)} + \frac{\lambda_1}{n+1}(n-1-\lambda_1)$$

so that $0 \geq \frac{\lambda_1}{n+1}\left(\frac{1}{n}\lambda_1 + (n-1) - \lambda_1\right) = \frac{\lambda_1}{n+1}\left(n-1-\left(\frac{n-1}{n}\right)\lambda_1\right) = \frac{n-1}{n+1}\cdot\lambda_1\left(1-\frac{\lambda_1}{n}\right)$

Since $\lambda_1 > 0$, it follows that $\lambda_1 \geq n$.

In the equality case, all inequalities become equalities. In particular, we get that

$$|\nabla^2 f_1 - \tfrac{\Delta f_1}{n} g|^2 = 0 \Longrightarrow \nabla^2 f_1 + f_1 g = 0 \quad \text{on } M. \qquad (\star)$$

Step 1: $\phi := f_1^2 + |\nabla f_1|^2 \equiv 1$

Proof: Pick any $X \in TM$. Then

$$\nabla_X \phi = 2 \cdot (f_1 \nabla_X f_1 + \langle \nabla_X \nabla f_1, \nabla f_1 \rangle)$$
$$= 2(f_1 g(-,-) + \mathrm{Hess} f_1(-,-))(X, \nabla f_1) = 0$$

Thus, ϕ is constant, and by our normalizations on f_1, $\phi \equiv 1$.

Step 2: Let $p \in M$ be such that $f(p) = \max_M f = 1$. Then

$$\cos^{-1} f_1(x) = d(p,x) \quad \text{in } M \setminus \mathrm{Cut}(p)$$

Proof: In normal coords @ p, send out a radial geodesic $\gamma(t)$ from p. Let $u(t) = f_1(\gamma(t))$. Then by $(\star)$ and $f_1^2 + |\nabla f_1|^2 = 1$,

$$\begin{cases} u'' - u = 0 \\ u(0) = 1 \\ u'(0) = 0 \end{cases}$$

$u'(t) = \langle \nabla f_1(\gamma), \dot\gamma \rangle$
$u''(t) = \nabla_{\dot\gamma} \langle \nabla f_1(\gamma), \dot\gamma \rangle$
$= \nabla^2 f_1(\dot\gamma, \dot\gamma)$
$= f_1 g(\dot\gamma, \dot\gamma) = f_1 \circ \gamma = u$

⊕ So $f_1(\gamma(t)) = u(t) = \cos(t) = \cos(d(p, \gamma(t)))$. Thus, inside the cut-locus, $u(x) = \cos(d(p,x))$. Lets write $\begin{cases} f_1(r) = u(r) \\ r(x) = \mathrm{dist}(x,p). \end{cases}$

Step 3: Inside $M \setminus \mathrm{Cut}(p)$, M is a round sphere.

Proof: First compute that $\nabla^2 u \overset{\otimes}{=} -\cos(r)\, dr \otimes dr - \sin(r) \nabla^2 r$.

Since $\nabla^2 r(X,Y) = \mathrm{II}_{\Sigma_r}(X,Y)$ when $X, Y \in T\Sigma_r$ (Let Σ_r = r-level set), we have

$$\mathrm{II}_{\Sigma_r}(X,Y) \overset{\otimes}{=} -\frac{1}{\sin r} \nabla^2 u(X,Y) \overset{(\star)}{=} \frac{1}{\sin r} u(r)\, g(X,Y) \overset{\oplus}{=} \cot(r) g|_{X,Y}.$$

Now, on $M \setminus \mathrm{Cut}(p)$, $g = dr^2 + h_r$ for some smooth family h_r on S^{n-1}. By the line above, $\mathrm{II}_{\Sigma_r} = \cot(r) g|_{\Sigma_r} = \cot(r) h_r$ and in general, $\mathrm{II}_{\Sigma_r}(-,-) = \frac{1}{2} \partial_r (h_r)(-,-)$.

Thus $\frac{1}{2} \partial_r (h_r) = \cot(r) h_r$. By ansatz, we must have $h_r = c \sin^2(r) \bar h$ for some fixed metric $\bar h$ on S^{n-1}, and by Euclidean asymptotics as $r \searrow 0$, we have $c = 1$ and $\bar h = g_{sd}$. Thus, on $M \setminus \mathrm{Cut}(p)$, we have

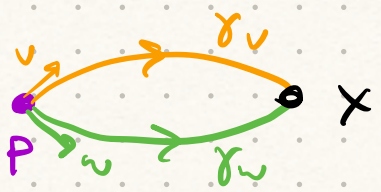
$$g = dr^2 + \sin^2(r) g_{sd} = g_{sd}$$

Step 4: Globalize by showing $\mathrm{Cut}_p = \{q\}$.

(a) Since $\sec \equiv 1$ on $M \setminus \mathrm{Cut}(p)$, no geodesic from p has any conjugate point before distance π (solve the Jacobi eqtn directly). Thus, $d(\exp_p) : B_\pi(0) \subseteq T_pM \to M$ is non singular, hence a local diffeo.

(b) To show $\exp_p : B_\pi(\vec{0}) \to M$ is injective:

Suppose $\exists\, v \neq w \in B_\pi(0)$ with $\exp_p(v) = \exp_p(w) = x$.



Let $r_v = |v|$, $r_w = |w|$. Then by the above,

$$f_1(x) = f_1(\gamma_v(r_v)) = \cos(r_v)$$
$$= f_1(\gamma_w(r_w)) = \cos(r_w)$$

So $r_v = r_w \equiv r$ (as $0 < r_v, r_w < \pi$). Thus, consider

$$\exp_p : \partial B_r(\vec{0}) \subseteq T_pM \longrightarrow \partial B_r(p)$$

This map is a local diffeo, $\partial B_r(\vec{0})$ is compact, and $\partial B_r(\vec{0})$, $\partial B_r(p)$ have the same dimension. Thus,

$\exp_p|_{\partial B_r(\vec{0})} : \partial B_r(\vec{0}) \to \partial B_r(p)$ is a covering map.

Moreover, as $\exp_p : B_\pi(0) \to M \setminus \mathrm{Cut}\{p\}$ is a local diffeo, the number of sheets in these coverings is locally constant in r. As $r \searrow 0$, $\exp_p$ becomes a true diffeo by local Euclidean-ness, hence this is a 1-sheeted covering and $\exp_p : B_\pi(0) \to M$ is a diffeo onto its image and $v = w$ ↘.

(c) Thus, $\exp_p(B_\pi(\vec{0})) \subseteq M \setminus \mathrm{Cut}\{p\}$. In particular, any $q \in \mathrm{Cut}\{p\}$ is distance π from p (max'l dist from p) and so

$$f_1(q) = \cos(\pi) = -1 = \min_M f_1$$

But critical points of f_1 are nondegenerate by (*), hence isolated. Thus, $\mathrm{Cut}\{p\}$ is discrete. However, as $\mathrm{cut}\{p\}$ is the image of $\partial B_\pi(\vec{0})$ under $\exp_p$, it must be connected. Thus $\mathrm{cut}\{p\} = \{q\}$.

On to the almost rigid regime:

Setup: (M^n, g) closed, $\mathrm{Ric} \geq (n-1)g$, $\varepsilon > 0$, and for some $k \geq 1$

$$n \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq n + \varepsilon.$$

Key Lemma 1: For $i = 1, \ldots, k$, $\forall p \geq 1$,

(A) $$f_i^2 + |\nabla f_i|^2 \leq 1 + \tau(\varepsilon; p)$$

(B) $$\| f_i^2 + |\nabla f_i|^2 - 1 \|_p \leq \tau(\varepsilon; p)$$

Proof: Use the Bochner Formula:

$$\Delta\left(f_i^2 + |\nabla f_i|^2\right) = 2\left[f_i \Delta f_i + |\nabla f_i|^2\right] + 2\left[|\nabla^2 f_i|^2 + \langle \nabla \Delta f_i, \nabla f_i \rangle + \mathrm{Ric}(\nabla f_i, \nabla f_i)\right]$$
$$= -2\lambda_i f_i^2 + 2|\nabla f_i|^2 + 2|\nabla^2 f_i|^2 - 2\lambda_i |\nabla f_i|^2 + 2\mathrm{Ric}(\nabla f_i, \nabla f_i)$$

Now introduce the comparisons

(i) $\mathrm{Ric}(\nabla f_i, \nabla f_i) \geq (n-1)|\nabla f_i|^2$

(ii) $$\left|\nabla^2 f_i - \frac{\Delta f_i}{n} g\right|^2 = |\nabla^2 f_i|^2 - \frac{(\Delta f_i)^2}{n} = |\nabla^2 f_i|^2 - \frac{\lambda_i^2}{n} f_i^2$$

We obtain

$$\Delta\left(f_i^2 + |\nabla f_i|^2\right) = 2\left[|\nabla^2 f_i|^2 - \frac{\lambda_i^2}{n} f_i^2\right] + 2\left[\mathrm{Ric}(\nabla f_i, \nabla f_i) - (n-1)|\nabla f_i|^2\right]$$
$$-2\lambda_i f_i^2 + (2 - 2\lambda_i)|\nabla f_i|^2 + 2\frac{\lambda_i^2}{n} f_i^2 + 2(n-1)|\nabla f_i|^2$$
$$= 2\left|\nabla^2 f_i - \frac{\Delta f_i}{n} g\right|^2 + 2\left[\mathrm{Ric}(\nabla f_i, \nabla f_i) - (n-1)|\nabla f_i|^2\right]$$
$$+ 2(\lambda_i - n)\left[\frac{\lambda_i}{n} f_i^2 - |\nabla f_i|^2\right]$$
$$\geq 2(\lambda_i - n)\left[\frac{\lambda_i}{n} f_i^2 - |\nabla f_i|^2\right] \qquad \text{(i), (ii)}.$$

Claim: f_i^2 and $|\nabla f_i|^2$ are bounded solely in terms of $\varepsilon > 0$.

Proof: By a Gradient Estimate (see Appendix), $|\nabla f_i|^2 \leq C_\varepsilon$. Since $\operatorname{avg} f_i^2 = 1$ and $\operatorname{diam} M \leq \pi$, f_i^2 is bounded as well. ∎

Therefore, $$\Delta(f_i^2 + |\nabla f_i|^2) \geq -\tau(\varepsilon : n) \qquad (*)$$

for $\varepsilon > 0$ small.

(A) Apply the Generalized Max. Princ. (see Appendix) to obtain

$$0 \leq f_i^2 + |\nabla f_i|^2 \leq \fint_M (f_i^2 + |\nabla f_i|^2) + \tau(\varepsilon : n)$$
$$= \frac{1}{n+1} + \frac{\lambda_i}{n+1} + \tau(\varepsilon : n)$$
$$\leq 1 + \tau(\varepsilon : n)$$

(B) Observe that $$1 \leq \operatorname{avg}(f_i^2 + |\nabla f_i|^2) = \frac{\lambda_i + 1}{n+1} \leq 1 + \tau(\varepsilon : n)$$

and $$f_i^2 + |\nabla f_i|^2 \leq 1 + \tau(\varepsilon : n)$$

so that $\| f_i^2 + |\nabla f_i|^2 - 1 \|_1 \leq \tau(\varepsilon : n)$. Indeed, let $\phi = f_i^2 + |\nabla f_i|^2$ and $\alpha = \operatorname{avg}_M \phi$. Then

$$0 = \fint_M (\phi - \alpha) = \underbrace{\frac{1}{\operatorname{vol} M} \int_{\{\phi \geq \alpha\}} (\phi - \alpha)}_{\leq \tau(\varepsilon : n)} + \underbrace{\frac{1}{\operatorname{vol} M} \int_{\{\phi < \alpha\}} (\phi - \alpha)}_{\geq -\tau(\varepsilon : n)}$$

hence $\fint_M |\phi - 1| \leq |1 - \alpha| + \fint_M |\phi - \alpha| \leq \tau(\varepsilon : n)$. Thus, for any $p \geq 1$,

$$\| f_i^2 + |\nabla f_i|^2 - 1 \|_p \leq \| |f_i^2 + |\nabla f_i|^2 - 1|^{p-1} \|_\infty^{\frac{1}{p}} \| f_i^2 + |\nabla f_i|^2 - 1 \|_1^{\frac{1}{p}} \leq \tau(\varepsilon : n, p)$$

As an important generalization of this result, we can extend these estimates to certain "spherical linear combinations" of eigenfunctions:

Lemma 2: Let $\phi = \sum_{i=1}^k \alpha_i f_i$ with $\left| \sum_{i=1}^k \alpha_i^2 - 1 \right| \leq \tau(\varepsilon : n)$.

Then

(A) $\phi^2 + |\nabla \phi|^2 \leq 1 + \tau(\varepsilon : n)$

(B) $\| \phi^2 + |\nabla \phi|^2 - 1 \|_p \leq \tau(\varepsilon : n, p) \quad \forall p \geq 1.$

Proof: The proof is largely the same as for the Key Lemma above, once we use orthogonality of the $f_1, \dots, f_{n+1}$.

Step 1: First check $$f_M \phi = 0$$

Step 2: Next,

$$\left| f_M \phi^2 - \frac{1}{n+1} \right| = \left| \sum_{i=1}^n \alpha_i \alpha_j f_M f_i f_j - \frac{1}{n+1} \right| = \left| \sum_{i=1}^n \alpha_i^2 f_M f_i^2 - \frac{1}{n+1} \right|$$

$$= \frac{1}{n+1} \left| \sum_{i=1}^k \alpha_i^2 - 1 \right| \leq \tau(\varepsilon : n)$$

Step 3: Similarly,

$$\left| f_M |\nabla \phi|^2 - \frac{n}{n+1} \right| = \left| \sum_{i=1}^k \alpha_i \alpha_j f_M \langle \nabla \phi_i, \nabla \phi_j \rangle - \frac{n}{n+1} \right| = \left| \sum_{i=1}^k \alpha_i^2 f_M |\nabla \phi_i|^2 - \frac{n}{n+1} \right|$$

$$= \frac{1}{n+1} \left| \sum_{i=1}^k \alpha_i^2 \lambda_i - n \right|$$

$$= \frac{1}{n+1} \left| \left(\sum_{i=1}^k \alpha_i^2 - 1 \right) \lambda_i + (\lambda_i - n) \right| \leq \tau(\varepsilon : n)$$

Step 4: We compute

here, a function which could be < 0

$$\Delta \phi = -\sum_{i=1}^k \alpha_i \lambda_i f_i = -\sum_{i=1}^k \alpha_i n f_i - \sum_{i=1}^k \alpha_i (\lambda_i - n) f_i = -n\phi + \widetilde{\tau(\varepsilon : n)}$$

using that $0 \leq f_i^2 + |\nabla f_i|^2 \leq 1 + \tau(\varepsilon : n)$. Thus, by Bochner,

$$\frac{1}{2} \Delta (\phi^2 + |\nabla \phi|^2) = \phi \Delta \phi + |\nabla \phi|^2 + |\nabla^2 \phi|^2 + \langle \nabla \Delta \phi, \nabla \phi \rangle + Ric(\nabla \phi, \nabla \phi)$$

$$= -n\phi^2 - \tau(\varepsilon : n) + |\nabla \phi|^2 + |\nabla^2 \phi|^2 + \langle \nabla \Delta \phi, \nabla \phi \rangle + Ric(\nabla \phi, \nabla \phi)$$

note that $|\nabla \phi|$

$$= -\frac{(\Delta \phi)^2}{n} - \tau(\varepsilon : n) + \text{“}\quad\text{”}$$

$$= \left(|\nabla^2 \phi|^2 - \frac{(\Delta \phi)^2}{n} \right) + |\nabla \phi|^2 - n|\nabla \phi|^2 + Ric(\nabla \phi, \nabla \phi) - \tau(\varepsilon : n)$$

$$= \left(|\nabla^2 \phi|^2 - \frac{(\Delta \phi)^2}{n} \right) + \left(Ric(\nabla \phi, \nabla \phi) - (n-1)|\nabla \phi|^2 \right) - \tau(\varepsilon : n)$$

$$= \tau(\varepsilon : n) \quad \text{(again, here a function with } \|\tau(\varepsilon : n)\|_\infty \leq \varepsilon \text{)}.$$

Step 5:

So, by the Generalized Maximum Princ. and the same finishing arguments in the Key Lemma, using Steps 2, 3, 4 we conclude that

$$0 \leq \phi^2 + |\nabla \phi|^2 \leq 1 + \tau(\varepsilon : n)$$

and $$\| \phi^2 + |\nabla \phi|^2 - 1 \|_p \leq \tau(\varepsilon : n, p) \quad \forall p \geq 1.$$

∎

Recall the map $\Phi: M \to \mathbb{R}^{n+1}$ define by $\Phi(x) = (f_1(x), \dots, f_{n+1}(x))$, which is supposed to approximate the identity $\Phi(x) = (x^1, \dots, x^{n+1})$ in the model case.

It is convenient here to define

$$\Phi_k : M \to \mathbb{R}^k \quad \text{by} \quad \Phi_k(x) = (f_1(x), \dots, f_k(x))$$

as the 1st k-coords of Φ (recall we are assuming here that $\lambda_1, \dots, \lambda_k$ are the eigenvalues in $[n, n+\varepsilon]$).

Denote by $d\Phi_k : TM \to \mathbb{R}^k$ the derivative acting as

$$d\Phi_k(v) = \begin{pmatrix} - \nabla f_1 - \\ \vdots \\ - \nabla f_k - \end{pmatrix}(v) = \begin{pmatrix} \langle \nabla f_1, v \rangle \\ \vdots \\ \langle \nabla f_k, v \rangle \end{pmatrix}$$

Lemma 3: (A first "mapping result")

(A) $|\Phi_k| \le 1 + \tau(\varepsilon : n)$

(B) $\| d\Phi_k \|_{T_x M, \mathbb{R}^k} \le 1 + \tau(\varepsilon : n) \quad \forall x \in M.$

Proof: (A) If $\Phi_k(x) = \vec{0}$ then is nothing to show, so assume $\Phi_k(x) = (f_1(x), \dots, f_k(x)) \neq \vec{0}$. Then choose

$$a_i = \frac{f_i(x)}{\left(\sum_{i=1}^k f_i(x)^2\right)^{\frac{1}{2}}} \qquad \left(\text{note } \sum_{i=1}^k a_i^2 = 1\right)$$

and apply Lemma 2 to $\phi(y) = \sum_{i=1}^k a_i f_i(y) = \sum_{i=1}^k \frac{f_i(x) f_i(y)}{\left(\sum_{i=1}^k f_i(x)^2\right)^{\frac{1}{2}}}$ to get that

$$\Phi^2 \le \phi^2 + |\nabla \phi|^2 \le 1 + \tau(\varepsilon : n).$$

In particular, at x,

$$|\Phi_k(x)|^2 = \sum_{i=1}^k f_i(x)^2 = \left[\frac{\left(\sum_{i=1}^k f_i(x)^2\right)}{\left(\sum_{i=1}^k f_i(x)^2\right)^{1/2}}\right]^2 = \left(\sum_{i=1}^k a_i f_i(x)\right)^2$$

$$= \phi^2(x) \le 1 + \tau(\varepsilon : n).$$

(B) Let $v \in UT_xM$, and suppose $d\Phi_k(v) \neq \vec{0}$. Choose

$$\beta_i = \frac{df_i(v)}{\left(\sum_{i=1}^k |df_i(v)|^2\right)^{\frac{1}{2}}} \qquad \left(\text{note } \sum_{i=1}^k \beta_i^2 = 1\right)$$

and apply Lemma 2 to $\phi(y) = \sum_{i=1}^k \beta_i f_i(y)$, so that $\nabla\phi(y) = \sum_{i=1}^k \beta_i \nabla f_i(y)$, to get

$$|\nabla\phi|^2 \leq \phi^2 + |\nabla\phi|^2 \leq 1 + \tau(\varepsilon:n).$$

In particular, at $v \in UT_xM$,

$$|d\Phi_k(v)|^2 = \sum_{i=1}^k |df_i(v)|^2 = \left(\frac{\sum_{i=1}^k |df_i(v)|^2}{\left(\sum_{i=1}^k |df_i(v)|^2\right)^{\frac{1}{2}}}\right)^2 = \left(\sum_{i=1}^k \beta_i\, df_i(v)\right)^2 = \langle \nabla\phi(x), v\rangle^2$$
$$\leq |\nabla\phi(x)|^2 \leq 1 + \tau(\varepsilon:n).$$

Thus, $\|d\Phi_k\|_{T_xM \to \mathbb{R}^k} \leq 1 + \tau(\varepsilon:n)$. ▨

At this point, we proceed in two sections to prove the Main Theorem. In this section, we show that:

Main Theorem ($\Longrightarrow$) $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon, n) > 0$ s.t.:

For any closed (M^n, g) with $\mathrm{Ric}_g \geq (n-1)g$: $\mathrm{rad}(M) \geq \pi - \delta \Rightarrow \lambda_{n+1} \leq n + \varepsilon$.

We need the following estimates of Colding (proved only using Bishop-Gromov)

Lemma 4: (Colding '96, Lemma 1.10) $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon, n) > 0$ s.t.:

For any closed (M^n, g) with $\mathrm{Ric}_g \geq (n-1)g$: if some distance function $d(x_0, -)$ is such that $\exists x \in M$ with $d(x_0, x) \geq \pi - \delta$, then $\cos d_{x_0}(\cdot)$ satisfies:

(A) $|f_M \cos d| \leq \tau(\varepsilon:n)$

(B) $\left|f_M |\cos d|^2 - \frac{1}{n+1}\right| \leq \tau(\varepsilon:n)$

(C) $\left\|\cos d - \sum_{i=1}^k a_i f_i\right\|_2 \leq \tau(\varepsilon:n)$

(D) $\left|\sum_{i=1}^k a_i^2 - 1\right| \leq \tau(\varepsilon:n)$

} here $\cos d = \sum_{i=0}^\infty a_i f_i$ is the Fourier expansion of $\cos d$.

Remark: Here we witness the first justifications that in the perturbative setting, $\cos d_p$ behaves similarly to a linear combo of eigenfunctions (in particular, the "k-jet" of its Fourier series). In the next lemma Petersen upgrades this result:

Lemma 5: In addition to the above, we also have

$$\left| \cos d_{x_0} - \sum_{i=1}^{k} a_i f_i \right| \leq \tau(\varepsilon : n).$$

Proof: By Lemma 3, $f := \sum_{i=1}^{k} a_i f_i$ has $\|f\|_\infty \leq 1 + \tau(\varepsilon : n)$.

Since $\|\cos d\|_\infty \leq 1$, we have that

$$\| f - \cos d \|_p^p \leq \| |f - \cos d|^{p-2} \|_\infty \| f - \cos d \|_2^2 \leq \tau(\varepsilon : n).$$

Moreover, in the global barrier sense, $\Delta d \leq (n-1)\cot d$, so

$$\begin{aligned} \Delta \cos d = d\nu(-\sin d \nabla d) &= -\sin d \, \Delta d - \cos d |\nabla d|^2 \\ &\geqslant -(n-1)\cos d - \cos d = -n \cos d \end{aligned}$$

so that (in the barrier sense)

$$\begin{aligned} \Delta(f - \cos d) &= -\sum_{i=1}^{k} a_i \lambda_i f_i - \Delta \cos d \\ &\leq -\sum_{i=1}^{k} a_i \lambda_i f_i + n \cos d \\ &= -\sum_{i=1}^{k} a_i (\lambda_i - n) f_i - n(f - \cos d) \\ &= \phi(x) + \tau(\varepsilon : n) \end{aligned}$$

Here $\phi(x) = n(f - \cos d)$ has $\|\phi\|_p \leq \tau(\varepsilon : n, p)$ by the above.

Fix now $p > n/2$, and apply the Generalized Max'm Princ. to obtain

$$\begin{aligned} f - \cos d &\geqslant \underset{M}{f}(f - \cos d) - C(n,p)\left(\|\phi\|_p + \left\| \sum_{i=1}^{k} a_i (\lambda_i - n) f_i \right\|_p \right) \\ &\geqslant -\tau(\varepsilon : n, p). \end{aligned} \tag{1}$$

Now feed this back into the above bounds to also get

$$\Delta(f - \cos d) \leq \tau(\varepsilon : n, p)$$

Since

- $|f(f - \cos d)| \leq \tau(\varepsilon : n) \Rightarrow |f - \cos d| \leq \tau(\varepsilon : n)$ at some point.
- $|\nabla(f - \cos d)| \leq 2 + \tau(\varepsilon : n)$

it follows then from the Albresch-Gromoll Max'm Princ. that we also have

$$f - \cos d \leq \tau(\varepsilon : n) \qquad (2)$$

everywhere.

Pairing (1) and (2), we have $|f - \cos d| \leq \tau(\varepsilon : n)$ (Supposing $p > n/2$).

everywhere on M. ▨

Main Theorem ($\Longrightarrow$) $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon, n) > 0$ s.t.:

For any closed (M^n, g) with $\mathrm{Ric}_g \geq (n-1)g$: $\mathrm{rad}(M) \geq \pi - \delta \Rightarrow \lambda_{n+1} \leq n + \varepsilon$.

Proof: If $\mathrm{rad}\, M \geq \pi - \delta(\varepsilon : n)$ with $\delta(\varepsilon : n)$ as in Lemma 4, 5, then the conclusions of those lemmas hold for all distance functions d_{x_0}.

In particular, we have that $|\cos d - f| = |\cos d_{x_0} - \sum_{i=1}^{k} \alpha_i f_i(x)| \leq \tau(\varepsilon : n)$ and $|\sum_{i=1}^{k} \alpha_k^2 - 1| \leq \tau(\varepsilon : n)$, where the α_k are the Fourier coefficients of $\cos d_{x_0}$:

$$\alpha_i = (n+1) \int_M f_i(x) \cos d_{x_0}(x)\, dx$$

In the rigid case, $\alpha_i = x_0^i$ (the i^{th} coordinate of the basepoint of $\cos d_{x_0}$), for $i = 1, \ldots, n+1$, so that the map

$$\Psi_{n+1}(x_0) = (\alpha_1, \ldots, \alpha_{n+1}) = (x_0^1, \ldots, x_0^{n+1}) = x_0.$$

In the almost rigid case, we so far know that

$\Psi_k(x_0) = (\alpha_1, \ldots, \alpha_k)$ has $\big||\Psi_k(x_0)|^2 - 1\big| = |\sum_{i=1}^{k} \alpha_i^2 - 1| \leq \tau(\varepsilon : n)$.

Let's compare this map to the map of eigenfunctions

$$\Phi_k(x_0) = (f_1(x_0), \ldots, f_k(x_0))$$

By Lemma 3, we know that $|\Phi_k(x)| \leq 1 + \tau(\varepsilon : n)$. By comparing Φ_k to Ψ_k, we can upgrade this to $\big||\Phi_k(x)|^2 - 1\big| \leq \tau(\varepsilon : n)$:

$$|\Psi_k(x_0) - \Phi_k(x_0)|^2 = \sum_{i=1}^k |a_i - f_i(x_0)|^2$$

$$= \sum_{i=1}^k |a_i|^2 + \sum_{i=1}^k |f_i(x_0)|^2 - 2f(x_0) \quad (f = \textstyle\sum_{i=1}^k a_i f_i)$$

$$\leq 2 + \tau(\varepsilon:n) - 2\cos d_{x_0}(x_0) + \tau(\varepsilon:n) \quad \text{(Lemma 5)}$$

$$= \tau(\varepsilon:n)$$

Thus, we have that

$$\left| |\Phi_k(x_0)|^2 - 1 \right| \leq \tau(\varepsilon:n)$$

and this implies that

$$\frac{k}{n+1} = \fint_M \sum_{i=1}^k f_i^2 = \fint_M |\Phi_k|^2 = 1 + \tau(\varepsilon:n)$$

Thus, we see that we must have $k = n+1$ provided that $\mathrm{rad}(M) \gtrsim \pi$.

Before we prove the converse, we establish one more lemma, establishing as above that $\Phi_{n+1}: M \to \mathbb{R}^{n+1}$ maps close to S^n. Unlike in the $\Rightarrow$ direction, we do not yet know $|\cos d - f| \leq \tau$, so we can't just apply Lemma 5.

For the rest of the notes, let (M^n, g) be a closed Riemannian manifold with $\mathrm{Ric}_g \geq (n-1)g$ and $\lambda_{n+1} \leq n + \varepsilon$.

Lemma 6: $\left| |\Phi_{n+1}|^2 - 1 \right| \leq \tau(\varepsilon:n)$

Proof: By normalization,

$$\fint_M |\Phi_{n+1}|^2 = \fint_M \sum_{i=1}^{n+1} f_i(x)^2 = 1.$$

We will upgrade this using the generalized maximum principle.

By Lemma 3, we also know that

$$|\Phi_{n+1}(x)|^2 \leq 1 + \tau(\varepsilon:n)$$

And so, as before,

$$\| |\Phi_{n+1}|^2 - 1 \|_p \leq \tau(\varepsilon : n, p)$$

for any $p \geq 1$. Next compute

$$\Delta |\Phi_{n+1}|^2 = \Delta \sum_{i=1}^{n+1} f_i^2 = \sum_{i=1}^{n+1} \left(2|\nabla f_i|^2 - 2\lambda_i f_i^2 \right)$$

(introduce $\lambda_i - n$)
$$= \sum_{i=1}^{n+1} \left(2|\nabla f_i|^2 - 2(\lambda_i - n) f_i^2 - 2n f_i^2 \right)$$

(introduce $f_i^2 + |\nabla f_i|^2$)
$$= \sum_{i=1}^{n+1} \left\{ 2\left(f_i^2 + |\nabla f_i|^2 \right) - 2(\lambda_i - n) f_i^2 - 2(n+1) f_i^2 \right\}$$

(use estimates on $f_i^2 + |\nabla f_i|^2 \approx 1$)
$$= \sum_{i=1}^{n+1} \left\{ 2\left(f_i^2 + |\nabla f_i|^2 - 1 \right) - 2(\lambda_i - n) f_i^2 + \left(2 - 2(n+1) f_i^2 \right) \right\}$$

$$= 2(n+1) \underbrace{\left[1 - \sum_{i=1}^{n+1} f_i^2 \right]}_{\left(1 - |\Sigma_{n+1}|^2\right)} + \sum_{i=1}^{n+1} 2(n - \lambda_i) f_i^2 + \sum_{i=1}^{n+1} 2\left(f_i^2 + |\nabla f_i|^2 - 1 \right)$$

Using all of the estimates we have collected so far, we see that

$$-\phi \leq \Delta |\Phi_{n+1}|^2 \leq \phi$$

for $\phi \in L^p(M)$ with any $p \geq 1$, and hence by the Generalized Maximum Principle we see that

$$\left| |\Phi_{n+1}|^2 - 1 \right| \leq \tau(\varepsilon : n)$$



Main Theorem ($\Leftarrow$) $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon, n) > 0$ s.t.:
For any closed (M^n, g) with $\mathrm{Ric}_g \geq (n-1)g$: $\lambda_{n+1} \leq n + \varepsilon \Rightarrow \mathrm{rad}(M) \geq \pi - \delta$.

Proof: By lemma 6 above, $\left| |\Phi_{n+1}|^2 - 1 \right| \leq \tau(\varepsilon : n)$.

Fix any $x_0 \in M$, and set $\alpha_i = f_i(x_0)$ and define

$$f = \sum_{i=1}^{n+1} \alpha_i f_i .$$

f is close to maximal at x_0 (as $f(x_0) = |\Phi_{n+1}(x_0)|^2 \geq 1 - \tau$)

Using lemma 2 (which says $|f|^2 + |\nabla f|^2 \leq 1 + \tau$), by decreasing each coefficient α_i a little we can arrange that for our $f(x)$ above,

$$|f|^2 + |\nabla f|^2 \leq 1$$

and

$$f(x_0) \geq 1 - \tau(\varepsilon : n) .$$

Claim: $|f - \cos d_{x_0}| \leq \tau(\varepsilon : n)$

Proof: Note that, by the above bounds, $\cos^{-1} f(x)$ is well defined and satisfies

$$|\nabla(\cos^{-1} f)| = \frac{|\nabla f|}{\sqrt{1 - f^2}} \leq 1$$

and

$$0 \leq \cos^{-1} f(x_0) \leq \tau(\varepsilon : n)$$

since $f(x_0) \approx 1$. Therefore,

$$\cos^{-1} f(x) \leq \cos^{-1} f(x_0) + d(x_0, x) \leq \tau(\varepsilon : n) + d(x_0, x)$$

and taking cosines yields

$$f(x) \geq \cos d_{x_0}(x) - \tau(\varepsilon : n) .$$

By the same proof method of Lemma 5 (take $\Delta(f - \cos d_{x_0}) = \phi$ and use the generalized max. princ.), we obtain the pointwise estimate

$$\Delta(f - \cos d_{x_0}) \leq \tau(\varepsilon : n)$$

Since

$$-\tau(\varepsilon : n) \leq f(x_0) - \cos d_{x_0}(x_0) \leq 0$$

we can apply the Albresch-Gromoll max princ. (to $f - \cos d + \tau$, to be precise) to obtain the global pointwise upper bound

$$f(x) - \cos d_{x_0}(x) \leq \tau(\varepsilon : n)$$

and thus, altogether,

$$|f - \cos d_{x_0}| \leq \tau(\varepsilon : n)$$



Now we are in the home stretch!

Recalling that $f_M f = 0$, we see by the claim that, For every $x_0 \in M$,

$$\left| f_M \cos d_{x_0} \right| \leq \tau(\varepsilon : n)$$

Claim: For (M^n, g) closed with $\mathrm{Ric}_g \geq (n-1)g$, this implies that $\mathrm{rad}\, M \geq \pi - \tau(\varepsilon : n)$.

Proof: First, we set things up:

Fix $x_0 \in M$. By Bishop-Gromov, if $A(r) := \mathrm{Area}(\partial B_r(x_0))$ and $\bar{A}(r) := \mathrm{Area}(\partial \bar{B}_r \subseteq (\mathbb{S}^n, g_{rd}))$, we have that

$$1 \searrow \frac{A(r)}{\bar{A}(r)} \searrow \quad \text{as } r \nearrow.$$

Likewise, if $V(r) = \mathrm{vol}(B_r(x_0))$ and $\bar{V}(r) = \mathrm{vol}(\bar{B}_r \subseteq (\mathbb{S}^n, g_{rd}))$,

$$1 \searrow \frac{V(r)}{\bar{V}(r)} \searrow \quad \text{as } r \nearrow.$$

By the co-area formula (with level sets $d_{x_0}^{-1}(r) = \partial B_r(x_0)$), for any function on M we have

$$\int_{B_r(x_0)} f = \int_0^r \int_{\partial B_t(x_0)} f(\omega)\, d\mathcal{H}^{n-1}(\omega)\, dt.$$

and so for a radial function $f = f \circ d_{x_0}$ (as in $\cos d_{x_0}$), we have

$$\int_{B_r(x_0)} f = \int_0^r f(t) A(t)\, dt.$$

Lastly, let $R := \mathrm{rad}(x_0)$. By assumption, we have that

$$\left| f_M \cos d_{x_0} \right| \leq \tau(\varepsilon : n).$$

and we must show $R \geq \pi - \tau(\varepsilon : n)$.

Step 1: $R > \pi/2$ for all $\varepsilon > 0$ sufficiently small ($\varepsilon \leq c_n$)

Proof: Let (M^n, g), $p \in M$, be as in the statement.

On the one hand, if $R \leq \pi/2$, then since $\cos d_{x_0} \geq 0$ everywhere we have

$$\int_M \cos d_{x_0} = \frac{1}{V(R)} \int_{B_R(x_0)} \cos d_{x_0} \geq \frac{1}{V(R)} \int_{B_{R/2}(x_0)} \cos d_{x_0} \geq \cos(R/2) \frac{V(R/2)}{V(R)}$$

$$\geq \cos(R/2) \frac{\bar{V}(R/2)}{\bar{V}(R)} .$$

On the other hand, for $R \in [0, \pi/2]$,

$$\cos(R/2) \frac{\bar{V}(R/2)}{\bar{V}(R)} \geq \cos\left(\frac{\pi}{4}\right) \cdot \frac{\bar{V}(\pi/4)}{\bar{V}(\pi/2)} \equiv c_n > 0$$

Therefore, if $\varepsilon > 0$ is so small that $\tau(\varepsilon : n) < c_n$, then the above gives a contradiction if $R \leq \pi/2$.

Step 2: $R \geq \pi - \tau(\varepsilon : n)$ for $\varepsilon > 0$ sufficiently small.

Let $\varepsilon > 0$ be so small that **Step 1** ensures that $R > \pi/2$ (given any such (M^n, g), $p \in M$ as in the statement). Then we can write, using the co-area formula with level sets $d_p^{-1}(r) = \partial B_r(p)$,

$$\int_M \cos d_{x_0} = \frac{1}{V(R)} \int_0^R \cos(r) A(r) \, dr$$

$$= \frac{1}{V(R)} \left[\int_0^{\pi - R} \cos(r) A(r) \, dr + \int_{\pi - R}^{\pi/2} \cos(r) A(r) \, dr + \int_{\pi/2}^{R} \cos(r) A(r) \, dr \right]$$

$$= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III})$$

For (II), we use non-negativity and monotonicity to estimate

$$(\mathrm{II}) = \frac{1}{V(R)} \int_{\pi - R}^{\pi/2} \cos(r) \frac{A(r)}{\bar{A}(r)} \bar{A}(r) \, dr \geq \frac{1}{V(R)} \frac{A(\pi/2)}{\bar{A}(\pi/2)} \int_{\pi - R}^{\pi/2} \cos(r) \bar{A}(r) \, dr$$

For (III), we use non-positivity and monotonicity to estimate

$$(\mathrm{III}) = \frac{1}{V(R)} \int_{\pi/2}^{R} \cos(r) \frac{A(r)}{\bar{A}(r)} \bar{A}(r) \, dr \geq \frac{1}{V(R)} \frac{A(\pi/2)}{\bar{A}(\pi/2)} \int_{\pi/2}^{R} \cos(r) \bar{A}(r) \, dr$$

Moreover, by symmetry and oddness, we have that

$$(II)+(III) \geq \frac{1}{V(R)} \frac{A(\pi/2)}{\bar{A}(\pi/2)} \int_{\pi-R}^{R} \cos(r)\,\bar{A}(r)\,dr \equiv 0.$$

Therefore, again using that $R > \pi/2$,

$$\int_M \cos d_{x_0} = (I)+(II)+(III) \geq \frac{1}{V(R)} \int_0^{\pi-R} \cos(r)\,A(r)\,dr$$

$$\geq \cos(\pi-R)\,\frac{V(\pi-R)}{V(R)}$$

$$\geq \frac{\cos(\pi-R)\,\bar{V}(\pi-R)}{\bar{V}(R)}$$

$$\geq \omega_n^{-1} \cos(\pi-R)\,\bar{V}(\pi-R) \geq 0$$

Since $\cos(\pi-R)$ and $\bar{V}(\pi-R)$ are known, bounded functions which only go to 0 for $R \in (\pi/2, \pi]$ when $R \to \pi$, we thus see that we can find (more or less explicitly) $\varepsilon > 0$ so small that $\tau(\varepsilon : n) > 0$ forces $R \geq \pi - \tau(\varepsilon : n)$.

Consequently, for every $x_0 \in M$ we have

$$\mathrm{rad}(x_0) \geq \pi - \tau(\varepsilon : n)$$

and hence

$$\mathrm{rad}(M) \geq \pi - \tau(\varepsilon : n)$$

as desired.

Appendix: In this section (M^n, g) is a Riem. mfd.

Theorem (Generalized Max. Principle)

Let $u: M \to \mathbb{R}$ be such that $\Delta u \geq -f$ for some $f(x) \geq 0$.

Then for any $p > n/2$,

$$u(x) \leq C \cdot K \|f\|_p + \int_M u$$

where:

- C is the best $(2, \frac{2n}{n-2})$ Sobolev constant of M
  (uniform when $\operatorname{diam} M \leq D$, $\mathrm{Ric} \geq kg$ by Gallot '88)
- $K = K(n, p)$

Theorem: (Gradient Estimate)

Let $\Delta f + \lambda f = 0$ on M. Then

$$|\nabla f(x)| \leq K \cdot \|f\|_2$$

where

- $K = K(\bar{k}(p), C, \lambda)$ with C as above, $p > n/2$, and:

Def: $\bar{k}(p) :=$ L^p measure of how much Ricci curvature lies below $n-1$

Theorem (Albresch-Gromoll)

Let $\varepsilon, a > 0$, $p > n/2$. $\exists\, b = b(\varepsilon, a, n, p) > 0$ s.t.:

For any $u: B_R(x_0) \to [0, \infty]$ on M with $\bar{k}(p, R) \leq b$ satisfying: $u(x_0) < b\, G(R/2)$ (here, $G(x)$ is an auxilliary function having to do with the fundamental solution of Δ on M),

$|\nabla u| \leq a$, and $\Delta u \leq \phi$ with $\|\phi\|_{L^p} \leq b$, we have:

$$\|u\|_{L^\infty(B_{R_0}(x_1))} \leq \varepsilon.$$